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14. ABSTRACT
The discontinuous Galerkin (DG) method continues to maintain heightened levels of interest within the simulation community because of the discretization flexibility it provides. Although one of the fundamental properties of the DG methodology and arguably its most powerful property is the ability to combine highorder discretizations on an inter-element level while allowing discontinuities between elements, this flexibility generates a plethora of difficulties. First, by evolving more degrees of freedom, a more restrictive CFL condition is needed, which is computationally expensive. Secondly, the lack of continuity across element interfaces hampers visualization efforts as many visualization packages assume higher levels of continuity. However, DG also has the property of superconvergence. That is, it achieves essentially twice the usual convergence rate at specific points within the mesh, or in a special norm, allowing for faster convergence to a reasonable solution. The goal of this research is to exploit the inherent property of superconvergence in order to:(1) create a better pairing between DG solutions and the time-stepping method used, and(2)create filters that allow highly accurate visualization of DG data.

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EXPLOITING SUPERCONVERGENCE IN DISCONTINUOUS GALERKIN

METHODS FOR IMPROVED TIME-STEPPING AND VISUALIZATION

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1 February 2013 – 31 July 2016

Abstract
The discontinuous Galerkin (DG) method continues to maintain heightened levels of interest within the simulation community because of the discretization flexibility it provides. Although one of the fundamental properties of the DG methodology and arguably its most powerful property is the ability to combine high-order discretizations on an inter-element level while allowing discontinuities between elements, this flexibility generates a plethora of difficulties. First, by evolving more degrees of freedom, a more restrictive *cfl* condition is needed, which is computationally expensive. Secondly, the lack of continuity across element interfaces hampers visualization efforts as many visualization packages assume higher levels of continuity. However, DG also has the property of superconvergence. That is, it achieves essentially twice the usual convergence rate at specific points within the mesh, or in a special norm, allowing for faster convergence to a reasonable solution.
The goal of this research is to exploit the inherent property of superconvergence in order to (1) create a better pairing between DG solutions and the time-stepping method used, and (2) create filters that allow highly accurate visualization of DG data.

Status/Progress
AFOSR funding to support this research was obtained in February 2013. This funding was used to support Mr. Xiaozhou Li (TU Delft, February 2013 – July 2015), Mr. Daniel Frean (UEA, July 2013 – July 2016) and Ms. Julia Docampo (October 2013 – July 2016). Mr. Li successfully defended his thesis in July 2015. Mr. Frean and Ms. Docampo are expected to submit their thesis in October of 2016. Additionally, this funding was used to support collaborative visits related to this grant. Ms. Docampo visited the University of Utah to work with Prof. Robert M. Kirby in July of 2014 and February-March of 2016. Mr. Frean visited the University of Massachusetts – Dartmouth to work with Prof. Sigal Gottlieb in February-March of 2016.

This funding has allowed the PI to make several contributions over the lifetime of this grant:
- In the first year of this grant, the main accomplishments were: 1) demonstrating numerically that there is indeed a "sweet spot" for scaling of the Smoothness-Increasing Accuracy Conserving (SIAC) filter when applied to nonuniform meshes; 2) Theoretically and numerical demonstration of the *2k+1* order accuracy of the SIAC filter when applied to meshes with smoothly-varying mesh sizes; 3) Numerically demonstrating the usefulness of the boundary SIAC filter for nonlinear equations [4-

7,9,15-17,20]; and 4) Linking the known results for the eigenstructure analysis of the discontinuous Galerkin (DG) method for the upwind flux.

- In the second year of the grant, the main accomplishments were: 1) Establishing a more theoretical and numerical understanding of a computationally efficient scaling for the SIAC filter for nonuniform meshes [7]; 2) Establishing connections to signal processing and generalizing the SIAC kernel through quasi-interpolation [19]; 3) Introducing a Hexagon SIAC filter to reduce the kernel footprint; 4) Establishing the theoretical viability of the SIAC filter for nonlinear scalar hyperbolic conservation laws, which includes examining the role of divided difference estimates [18,25-26]; 5) Establishing superconvergence of Semi-Lagrangian DG (SLDG) methods [12]; as well as 6) Establishing theoretical and numerically that the upwind biased (non-monotonic) flux is more beneficial for even polynomial degree approximations than the upwind flux. This was used to computationally demonstrate the role of the flux in minimizing errors in long-time simulations [2,23].

- In the last year of the grant, there have been two major areas of research: 1) Fully discrete estimates for the dispersion and dissipation errors were obtained for various time-stepping strategies [3,24]; and 2) Establishing the optimal kernel footprint for the SIAC filter for computational efficiency [1,13,22]. The key findings for each of these areas are summarized in the next section.

Key Findings in final year of grant

- *Fully discrete estimates for the dispersion and dissipation errors.* For this study the following time-stepping schemes were paired with the DG scheme: the Runge-Kutta scheme of 2nd, 3rd and 4th orders (RK2, RK3, RK4), Lax-Wendroff (LW), New Lax-Wendroff (nLW) and Two Derivative Runge-Kutta (TDRK) schemes. For LW and TDRK, the dissipation and dispersion errors are only *k+1* order accurate. For the RK schemes and nLW scheme, the dispersion error is superconvergent of *2k+1*. The nLW scheme imposes an auxiliary formulation for the derivatives in order to recover the *2k+1* order accurate dispersion error. This technique was introduced for the TDRK (labelled nTDRK) and the superconvergent dispersion error was obtained. The equivalence in long time error behaviour for the RK schemes as well as nLW and nTDRK was established (Figure 1) [?].
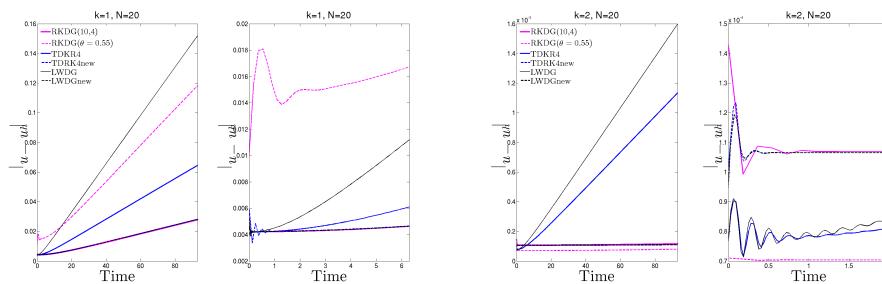

Figure 1: Time history plots of the DG spatial discretization paired with the Runge-Kutta (RK), Lax-Wendroff (LW) and Two Derivative Runge-Kutta (TDRK) schemes for *k=1* (Left two plots) and *k=2* (Right two plots). The solid blue and black lines represent the original LW and TDRK schemes. The dashed red line represents and upwind-biased flux within the DG scheme. For *k=1*, the upwind DG with RK is equivalent to nLW and nTDRK. For *k=2*, the upwind-biased flux paired with the Runge-Kutta scheme is optimal.

- *Establishing the optimal kernel footprint for the SIAC filter for computational efficiency.* This endeavour consisted of two key components: 1) the effect of rotating

the kernel; and 2) numerical and theoretical demonstration of the ability to use a one-dimensional SIAC kernel for multi-dimensional data. In computational experiments, it was noted that given the correct kernel scaling for the rotated kernel, superconvergence in the errors as well as reduction in the errors occurred. However, the computational efficiency of rotating the kernel hampered simulations because of the necessity to determine the intersections between the DG mesh and kernel mesh. When the theory was investigated, it was determined that a tensor product filter was not necessary for multi-dimensional data. The error between the exact solution and SIAC filtered solution remains

$$||u\text{-}u_h^*|| <= Ch^{2p+1}.$$

This presents a significant reduction in the kernel footprint as well as an increase in computational efficiency (Figure 2).
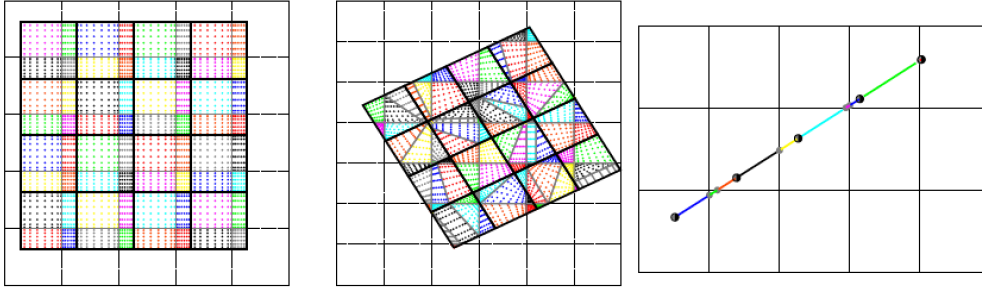

Figure 2: Illustrations of the DG mesh/Kernel mesh intersection regions. Each region requires implementation of a multi-dimensional quadrature rule. *Left:* Cartesian-coordinate aligned filter. *Center:* Rotated filter. *Right:* One-dimensional filter.

The one-dimensional kernel for two rotations, p/4 and 3p/4, was then tested on the DG approximation to a two-dimensional advection equation. This was compared with the traditional cartesian-aligned tensor product filter. Results demonstrated error reduction and superconvergence (Table 1). That is, the SIAC Line filter is able to improve the convergence rate of the DG approximation from *k+1* to *2k+1* while reducing the error. The contour plots for the errors are given in Figure 3.

N	L^2-error	Order	$\theta = 3\pi/4$ L^2-error	Order	$\theta = \pi/4$ L^2-error	Order	$\theta = 0$ L^2-error
			$\mathbb{P}^1$				
20	9.7e-03	-	1.5e-03	-	2.7e-03	-	1.6e-03
40	2.4e-03	2.02	1.9e-04	2.98	2.6e-04	3.33	2.0e-04
80	5.9e-04	2.01	2.4e-05	2.99	2.8e-05	3.21	
			$\mathbb{P}^2$				
20	2.4e-04	-	1.5e-06	-	1.4e-04	-	6.1e-06
40	2.9e-05	3.01	4.7e-08	4.99	2.3e-06	5.91	1.2e-07
80	3.6e-06	3.01	1.5e-09	5.00	3.7e-08	5.95	-
			$\mathbb{P}^3$				
20	4.5e-06	-	7.7e-10	-	1.6e-05	-	1.4e-07
40	2.8e-07	4.01	6.9e-12	6.79	6.9e-08	7.87	5.6e-10

Table 1: Results of the one-dimensional SIAC filter applied to the DG approximation to a 2D linear advection equation with sine initial condition. The left two columns represent the errors and order in the DG solution, the right column the error of the traditional tensor product SIAC filter and the middle four columns the errors and orders of the one-dimensional SIAC filter for two different rotations. The one-dimensional SIAC filter reduces the error and increases the order of accuracy from *k+1* to *2k+1*.
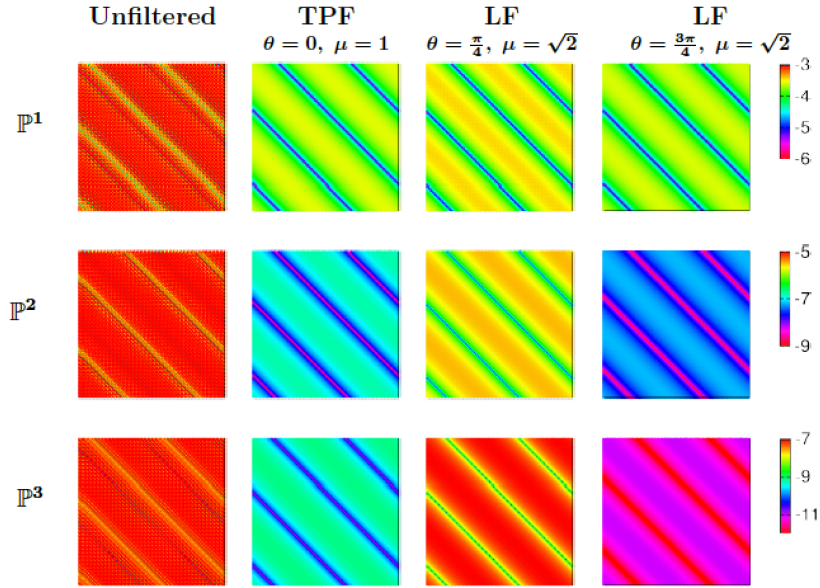
Figure 3: Contour plots of the error for the DG approximation (left), tensor product SIAC filter (center left), SIAC line filter with a rotation of *p/4* (center right), and SIAC line filter with a rotation of *3p/4* (right) for the 2D advection equation with sine initial condition.

Most Significant Accomplishments

Over the period of the grant, the most significant accomplishments are:

(1) Establishment of the relationship of the flux in the discontinuous Galerkin approximation to superconvergent dissipation and dispersion errors. This ensures that waves are correctly propagated forward in time, and thus ensuring a reduced long-time error [23-24].

(2) Establishing the SIAC line filter both computationally and theoretically. Using a one-dimensional filter for multi-dimensional data has a reduced computational cost that ensures error reduction as well as an improvement in the convergence rate. Additionally it ensures smoothness of the approximation in all directions [22].

(3) Establishing connections to signal processing by investigating the relation between SIAC filtering and quasi-interpolation [19].

(4) Establishment of a computationally efficient boundary SIAC filter. This allows for an increase of the smoothness of the DG approximation and generally a reduction in the error [15-17,20].

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References (Presentations and Publications)

Presentations
1. J. Docampo, "Geometry of rotated Smoothness-Increasing-Accuracy-Conserving (SIAC) filters", *The Mathematics of Finite Element Methods and Applications 2016.* Brunel University, United Kingdom. June 14-17, 2016.
2. D.J. Frean, "Superconvergence Properties of Discontinuous Galerkin Methods Based on Upwind-Biased Fluxes for Linear Hyperbolic Equations", *8th International Congress on Industrial and Applied Mathematics (ICIAM)*, Beijing, China. August 10-14, 2015.
3. D.J. Frean, "Superconvergence of discontinuous Galerkin methods for hyperbolic conservation laws : the numerical flux and dissipation/dispersion errors", *The Mathematics of Finite Element Methods and Applications,* Brunel University, United Kingdom. June 14-17, 2016.
4. X. Li, "Computationally Efficient Boundary Filtering using SIAC Methods", *Mathematics of Finite Elements and Applications (MAFELAP).* Brunel University, London, UK. 11-14 June 2013.
5. X. Li, "SIAC Filtering of DG Methods – Boundary and Nonuniform Mesh*", International Conference on Spectral and Higher Order Methods (ICOSAHOM),* University of Utah, Salt Lake City, Utah, USA. June 23-27, 2014.
6. J.K. Ryan, "Computationally Efficient Position-dependent SIAC Filtering for Nonlinear Hyperbolic Conservation Laws", *Second Workshop on the Development and Application of High-Order Numerical Methods.* Xiamen University, Xiamen, China. 18-21 May 2013.
7. J.K. Ryan, "Smoothness-Increasing Accuracy-Conserving (SIAC) Filtering for Discontinuous Galerkin Approximations to Nonlinear Hyperbolic Equations", *International conference on Applied Mathematics, Modeling and Computational Science*, Waterloo, Canada. 26-30 August 2013.
8. J.K. Ryan, "Exploiting Superconvergence through Smoothness-Increasing Accuracy-Conserving (SIAC) Filtering", *International Conference on Spectral and Higher Order Methods (ICOSAHOM),* University of Utah, Salt Lake City, Utah, USA. June 23-27, 2014.
9. J.K. Ryan, "Applications of Smoothness-Increasing Accuracy-Conserving (SIAC) Filtering for the discontinuous Galerkin Approximation to Nonlinear Hyperbolic Equations", *World Congress on Computational Mechanics XI,* Barcelona, Spain. July 20-25, 2014.
10. J.K. Ryan, "Exploiting Approximation Properties: Improving Filtering and Discontinuity Detection", *Mathematics and Statistics Departmental Seminar, University of Reading,* Reading, United Kingdom. October 21, 2014.
11. J.K. Ryan, "Superconvergence Extraction: Interpolation Versus Filtering", *Applied and Computational Mathematics seminar, University College – Dublin*, Dublin, Ireland. December 3, 2014.
12. J.K. Ryan, "Superconvergence and Superconvergent Filtering for Semi-Lagrangian discontinuous Galerkin (SLDG) Schemes", *8th International Congress on Industrial and Applied Mathematics (ICIAM 2015).* Beijing, China. August 20-14, 2015.
13. J.K. Ryan, "One-dimensional Filters for Multi-dimensional data", *International Conference on Spectral and Higher Order Methods (ICOSAHOM).* Rio de Janeiro, Brazil. June 27 - July 1, 2016.

Publications (Appeared or Accepted)
14. L.Ji and J.K. Ryan, "Smoothness-Increasing Accuracy-Conserving (SIAC) Filters in Fourier Space." *Proceedings of the 2014 Spectral and High Order Methods for Partial Differential Equations ICOSAHOM 2014,* Salt Lake City, Utah. Lecture Notes in Computational Science and Engineering, Springer, **vol. 106** (2015), pp 415--423.
15. X.Li and J.K. Ryan, "SIAC Filtering for Nonlinear Hyperbolic Equations." *Proceedings of the 2013 conference on Interdisciplinary Topics in Applied Mathematics, Modeling and Computational Science,* Waterloo, Canada. Springer Proceedings in Mathematics & Statistics, Springer, **vol. 117** (2015), pp 285--291.

16. X. Li, J.K. Ryan, R.M. Kirby and C. Vuik, "Smoothness-Increasing Accuracy-Conserving (SIAC) Filters for Derivative Approximations of discontinuous Galerkin (DG) Solutions over Nonuniform Meshes and Near Boundaries." *Journal of Computational and Applied Mathematics,* **vol. 294** (2016), pp. 275—296.
17. X. Li, J.K. Ryan, R.M. Kirby and C. Vuik, "Smoothness-Increasing Accuracy-Conserving (SIAC) Filters for Derivative Approximations of discontinuous Galerkin (DG) Solutions over Nonuniform Meshes and Near Boundaries." *Journal of Computational and Applied Mathematics,* **vol. 294** (2016), pp. 275--296.
18. X. Meng and J.K. Ryan, "Discontinuous Galerkin methods for nonlinear scalar hyperbolic conservation laws: divided difference estimates and accuracy enhancement." *Numerische Mathematik*, accepted.
19. M. Mirzargar, J.K. Ryan and R.M. Kirby, "Smoothness-Increasing Accuracy-Conserving (SIAC) Filtering and Quasi-Interpolation: A Unified View." *Journal of Scientific Computing,* **vol. 67** (2016), pp 237--261.
20. J.K. Ryan, X. Li, R.M. Kirby and C. Vuik, "One-Sided Position-Dependent Smoothness-Increasing Accuracy-Conserving (SIAC) Filtering Over Uniform and Non-Uniform Meshes." *Journal of Scientific Computing*, **vol. 64** (2015), pp 773--817.
21. J.K. Ryan, "Exploiting Superconvergence through Smoothness-Increasing Accuracy-Conserving (SIAC) Filtering." *Proceeding of the International Conference on Spectral and High Order Methods for Partial Differential Equations ICOSAHOM 2014*, Salt Lake City, Utah. Lecture Notes in Computational Science and Engineering, Springer, **vol. 106** (2015), pp 87--102.

Publications (Preprints and Submissions)
22. J. Docampo, J.K. Ryan, M. Mirzargar and R.M. Kirby, "Line Smoothness-Increasing Accuracy-Conserving (SIAC) Filters", preprint.
23. D.J. Frean and J.K. Ryan, "Superconvergence properties of an upwind-biased discontinuous Galerkin method." Submitted to *SIAM Journal on Numerical Analysis*.
24. D.J. Frean, J.K. Ryan and D. Seal, "Superconvergence of a two-derivative Runge-Kutta discontinuous Galerkin Method: The time evolution of dispersion/dissipation errors", preprint.
25. X. Meng and J.K. Ryan, "Divided difference estimates and accuracy enhancement of discontinuous Galerkin methods for nonlinear symmetric systems of hyperbolic conservation laws." Submitted to *IMA Journal of Numerical Analysis*.
26. X. Meng and J.K. Ryan, "Multi-dimensional divided difference estimates for discontinuous Galerkin methods for nonlinear scalar hyperbolic conservation laws", preprint.

Personnel Supported During Duration of Grant

Julia Docampo	Ph.D. student, University of East Anglia
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Jennifer K. Ryan	Lecturer, University of East Anglia

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Mahsa Mirzargar	Post-doctoral Student, University of Utah.
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